

Propagation of Signals in Nonlinear Transmission Lines

Abstract: This communication reports the results of a theoretical investigation of the propagation of signals in transmission lines whose nonlinear parameters are distributed capacitance and conductance. Such lines have their physical counterpart in lossless transmission lines that are loaded with closely spaced transistor bases or junction diodes, and in waveguides filled with certain ferroelectric ceramic materials. The analysis is carried out by conventional perturbation theory, and therefore the nonlinearities and dissipation are necessarily assumed to be small. The conditions for the formation of shock waves in such lines are also derived.

Introduction

In recent years, considerable interest has been shown in problems associated with the propagation of electromagnetic waves in nonlinear media in order to effectively utilize the nonlinear effects in both the radio and the optical range. When losses in the medium are small, shock wave formation is possible.¹⁻⁵ Thus, transmission lines or waveguides with nonlinear propagation media can be used for sharpening wavefronts and consequently for harmonic generation and parametric amplification of electromagnetic signals.⁶⁻⁹

The objectives of the present study are to obtain analytical results for the deformation of signals as they propagate in transmission lines whose nonlinear parameters are distributed capacitance and conductance. A physical counterpart of such a transmission system is a transmission line or a waveguide filled with certain ferroelectric ceramic materials.⁹ A lossless transmission line loaded with closely spaced transistor bases or junction diodes also can be considered, as a first approximation, to be a uniform transmission line with a nonlinear dielectric and a nonlinear conductor. The circuit representation for these two cases is shown in Fig. 1.

When the nonlinearities are large over the signal amplitudes, the problem does not always admit of analytical solution; one must resort to numerical or graphic methods. However, when nonlinearities are small, we can make use of conventional perturbation techniques to obtain analytical results. Khokholov² has used this method to study the formation of shock waves in lines with nonlinear dielectric and a linear loss mechanism. The present study on similar lines serves to extend these results by taking into account the

nonlinear nature of the shunt losses in the line.

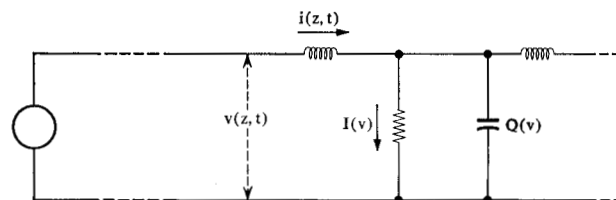


Figure 1 Semi-infinite transmission line with nonlinear distributed capacitance and conductance.

Formal solution of the problem

If $v(z, t)$ and $i(z, t)$ represent the voltage and current on the transmission line system shown in Fig. 1, their propagation can be described by the following equations:

$$\frac{\partial v}{\partial z} = -L \frac{\partial i}{\partial t}, \quad (1)$$

$$\frac{\partial i}{\partial z} = -\left[\frac{\partial Q(v)}{\partial t} + I(v)\right]. \quad (2)$$

Here $Q(v)$ is the charge on the capacitor, and the current $I(v)$ accounts for the shunt dissipation in the line.

Eliminating the current variable $i(z, t)$ from these equations, we obtain the following second-order partial differential equation describing the propagation of voltage $v(z, t)$ along the line:

$$\frac{\partial^2 v}{\partial z^2} - L \frac{\partial^2 Q(v)}{\partial t^2} - L \frac{\partial I(v)}{\partial t} = 0. \quad (3)$$

At the input to the line, the voltage is given as

$$v(z, t) = \varphi(t) \quad \text{at } z = 0. \quad (4)$$

Restricting ourselves to the case in which the line is semi-infinite in the z -direction, we look for solutions of Eq. (3) having the form of waves traveling to the right.

Both the charge Q on the capacitance and the shunt current $I(v)$ are nonlinear functions of the voltage. Since we shall consider only small nonlinearities and losses, the functions $Q(v)$ and $I(v)$ can be taken to be

$$Q(v) = C_0 v + \epsilon D v^2, \quad (5)$$

$$I(v) = \epsilon(G_0 v + F v^2), \quad (6)$$

where ϵ is the explicit small parameter, C_0 and G_0 are the fixed part of the capacitance and conductance per unit length and D and F are coefficients of the corresponding voltage-dependent parts.

If there is no attenuation and nonlinearity present, Eq. (3) has a solution of the form

$$v(z, t) = \varphi(t - z\sqrt{LC_0}), \quad (7)$$

and the propagation is without distortion. It is therefore reasonable to assume that, for small attenuation and nonlinearity, the solution of (3) does not differ drastically from (7) and, hence, we may assume a solution of the form

$$v = v(\epsilon z, t - z\sqrt{LC_0}). \quad (8)$$

Making the following transformation of variables:

$$\xi = t - z\sqrt{LC_0} = t - \frac{z}{u_0}, \quad (9)$$

$$x = \epsilon z, \quad (10)$$

we write the equation (3) as

$$\epsilon^2 \frac{\partial^2 v}{\partial x^2} - \frac{2\epsilon}{u_0} \frac{\partial^2 v}{\partial \xi \partial x} + \frac{1}{u_0^2} \frac{\partial^2 v}{\partial \xi^2} - L \frac{\partial}{\partial \xi} \left(C(v) \frac{\partial v}{\partial \xi} \right) - LG(v) \frac{\partial v}{\partial \xi} = 0, \quad (11)$$

where

$$C(v) = \frac{\partial Q}{\partial v} = C_0 + 2\epsilon D v, \quad (12)$$

$$G(v) = \frac{\partial I}{\partial v} = \epsilon(G_0 + 2Fv). \quad (13)$$

Simplifying (11) by substituting (12) and neglecting second-order terms in ϵ , we obtain

$$\frac{2}{u_0} \frac{\partial^2 v}{\partial \xi \partial x} + L \frac{\partial}{\partial \xi} \left(2Dv \frac{\partial v}{\partial \xi} \right) + LG(v) \frac{\partial v}{\partial \xi} = 0, \quad \text{or}$$

$$\frac{\partial}{\partial \xi} \left[\frac{2}{u_0} \frac{\partial v}{\partial x} + 2LDv \frac{\partial v}{\partial \xi} + LG_0 v + LFv^2 \right] = 0. \quad (14)$$

Integrating equation (14) over ξ , we have

$$\frac{\partial v}{\partial x} + 2\alpha Dv \frac{\partial v}{\partial \xi} + \beta(v + \lambda v^2) = h(x), \quad (15)$$

where

$$\alpha = \frac{u_0 L}{2},$$

$$\beta = \alpha G_0,$$

$$\lambda = \frac{F}{G_0},$$

$$\gamma = 2\alpha D, \quad (16)$$

and $h(x)$ is an arbitrary function of x . Since we are assuming that the line is in a quiescent state to start with, we can set $h(x) = 0$ and are left with a first-order quasi-linear partial differential equation:

$$\frac{\partial v}{\partial x} + \gamma v \frac{\partial v}{\partial \xi} + \beta(v + \lambda v^2) = 0. \quad (17)$$

This equation can now be considered to be the defining equation for wave propagation, and its solution and interpretation are the primary goals of the following discussion. It can be solved by the method of characteristics.¹⁰ The characteristic equations are

$$\frac{dx}{ds} = 1, \quad (18)$$

$$\frac{d\xi}{ds} = \gamma v, \quad (19)$$

$$\frac{dv}{ds} = -\beta(v + \lambda v^2), \quad (20)$$

where s is the parameter for the characteristic curves. Integrating these characteristic equations, we find

$$x = x_0 + s, \quad (21)$$

$$\xi = \xi_0 + \gamma \left[vs - \int s dv \right], \quad (22)$$

$$v = \frac{1}{e^{B_0 + \beta s} - \lambda}, \quad (23)$$

where x_0 , and ξ_0 , and B_0 are determined by initial and boundary conditions. Substituting (23) in (22), we have

$$\xi = \xi_0 + \gamma e^{-B_0} \left[\frac{s}{e^{\beta s} - \lambda e^{-B_0}} + \beta \int \frac{s e^{\beta s}}{(e^{\beta s} - \lambda e^{-B_0})^2} ds \right]. \quad (24)$$

Performing the integration in (24) and simplifying it, we have the following set of equations:

$$x = x_0 + s,$$

$$\xi = \xi_0 + \frac{\gamma}{\beta \lambda} \ln [1 - \lambda e^{-(B_0 + \beta s)}],$$

$$v = \frac{1}{e^{B_0 + \beta s} - \lambda}. \quad (25)$$

Identifying s with the space variable and setting $s = 0$ yields

$$\begin{aligned} x_0 &= 0, \\ \xi_0 &= t - \frac{\gamma}{\beta\lambda} \ln(1 - \lambda e^{-B_0}), \\ B_0 &= \ln \frac{1 + \lambda\varphi(t)}{\varphi(t)}. \end{aligned} \quad (26)$$

Substituting the values of ξ_0 and B_0 in (25) and simplifying it provides the following equation

$$\begin{aligned} \xi &= t - \frac{\gamma}{\beta\lambda} \ln(1 - \lambda e^{-B_0}) \\ &\quad - \frac{\gamma}{\lambda} \left[x - \frac{1}{\beta} \ln(e^{\beta x} - \lambda e^{-B_0}) \right]. \end{aligned} \quad (27)$$

Eliminating t from the equation with the help of (26) then yields the formal solution of the problem as an implicit relationship,

$$\begin{aligned} \xi &= \varphi^{-1} \left[\frac{e^{\beta x} v}{1 + \lambda v(1 - e^{\beta x})} \right] \\ &\quad - \frac{\gamma}{\beta\lambda} \ln[1 + \lambda v(1 - e^{\beta x})]. \end{aligned} \quad (28)$$

When the nonlinearity in the conductance is absent, $\lambda \rightarrow 0$; and then

$$\xi = \varphi^{-1}[ve^{\beta x}] + (\gamma/\beta)(e^{\beta x} - 1)v, \quad (29)$$

a result which is identical to that of Khokholov.²

Formation of shock waves

• Sinusoidal input

If the input signal is sinusoidal, then at $z = 0$,

$$v = v_0 \sin \omega t, \quad (30)$$

and the function φ^{-1} which is the inverse of the function φ of Eq. (4) is given by

$$\varphi^{-1}(v) = (1/\omega) \sin^{-1}(v/v_0). \quad (31)$$

Equation (28) now takes the following form:

$$\begin{aligned} \omega\xi &= \sin^{-1} \left[\frac{e^{\beta x} v/v_0}{1 + \lambda v(1 - e^{\beta x})} \right] \\ &\quad - \frac{\gamma\omega}{\beta\lambda} \ln[1 + \lambda v(1 - e^{\beta x})]. \end{aligned}$$

Letting

$$K = \gamma/\beta\lambda = 2D/F, \quad (32)$$

$$y = \frac{e^{\beta x} v/v_0}{1 + \lambda v(1 - e^{\beta x})}, \quad \text{and} \quad (33)$$

$$Z = \lambda v_0(1 - e^{-\beta x}), \quad (34)$$

we can write

$$\omega\xi = \sin^{-1} y + K\omega \ln(1 + Zy). \quad (35)$$

The distortion of the waveform as it propagates down the line can be observed by plotting (35) for various values of Z . Khokolov calls Z the "reduced distance"; it characterizes the extent to which the nonlinearity of the system manifests itself. Actually Z is just a one-to-one nonlinear transformation of the space variable x . It has been introduced mainly for the convenience of mathematical manipulation.

For a given value of v_0 and $K\omega$ the point of onset of shock waves can be obtained either from plots of y vs. $\omega\xi$ using (35) or by the following procedure. Since the onset of the shock wave corresponds to

$$d(\omega\xi)/dy = 0,$$

we have from equation (35) the following condition for this event:

$$-\frac{1}{(1-y^2)^{1/2}} + K\omega \frac{Z}{1+Zy} = 0.$$

Solving for Z , we have two possible solutions:

$$Z_1 = -\frac{1}{y + K\omega(1-y^2)^{1/2}}, \quad (36)$$

and

$$Z_2 = -\frac{1}{y - K\omega(1-y^2)^{1/2}}. \quad (37)$$

We must now find the value of y that yields a minimum value of Z . It can easily be shown that for

$$y_{1,2} = \pm (1 + K^2\omega^2)^{-1/2}, \quad (38)$$

$$\left. \frac{dZ_1}{dy} \right|_{y=y_{1,2}} = \left. \frac{dZ_2}{dy} \right|_{y=y_{1,2}} = 0,$$

and

$$\left. \frac{d^2 Z_1}{dy^2} \right|_{y=y_{1,2}} < 0,$$

but

$$\left. \frac{d^2 Z_2}{dy^2} \right|_{y=y_{1,2}} > 0.$$

Hence, substituting (38) in (37), we obtain two values of Z for the onset of shock waves.

$$Z_{2,1} = \frac{1}{(1 + K^2\omega^2)^{1/2}}, \quad (39)$$

$$Z_{2,2} = -\frac{(1 + K^2\omega^2)^{1/2}}{1 - K^2\omega^2}. \quad (40)$$

Hence, if $F > 0$, the shock waves start forming at a "reduced distance" corresponding to $Z = (1 + K^2\omega^2)^{-1/2}$. For $F < 0$, we have to look for the least negative value of Z . Proceeding exactly as before, we arrive at the conclusion that in this case the onset of shock waves is given by $Z = -(1 + K^2\omega^2)^{-1/2}$.

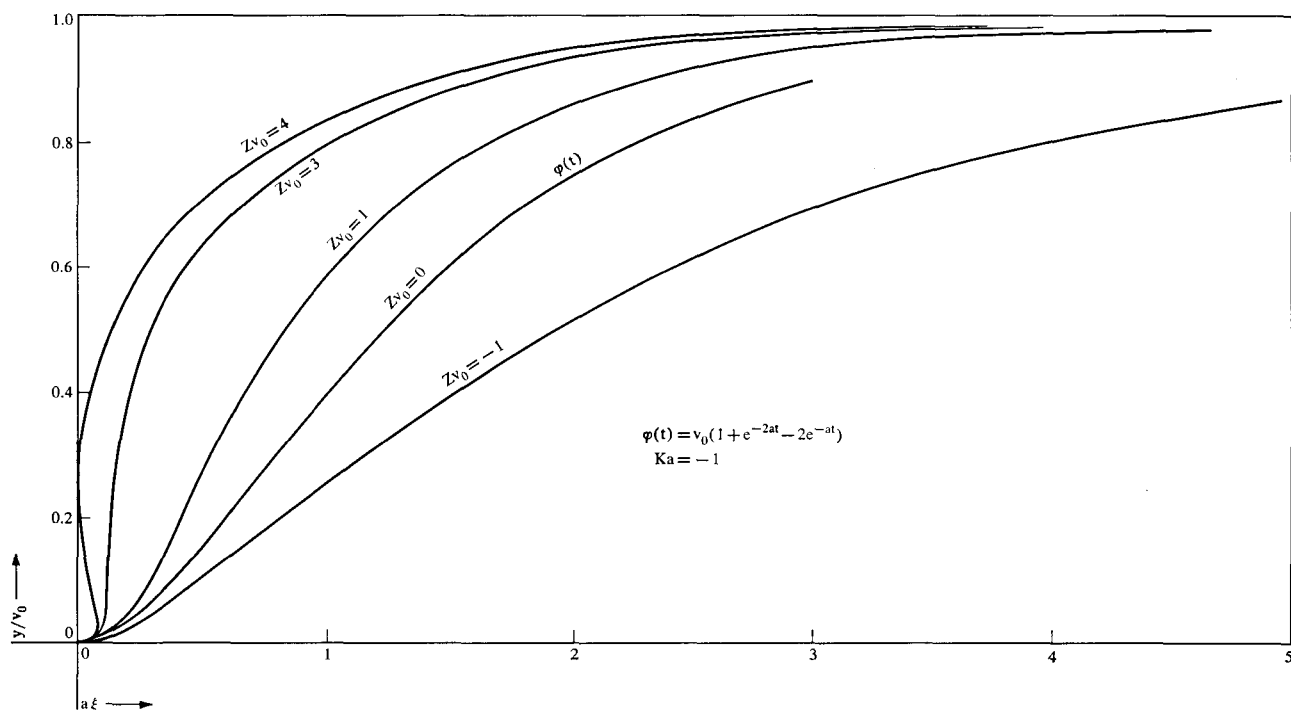


Figure 2 Formation of shock waves in nonlinear transmission lines.

If v_{crit} is defined as the amplitude of the input signal below which shock waves will not be formed in the line at any distance from the input, we can obtain its value from (39). For $F > 0$,

$$\begin{aligned} v_{crit} &= Z_{2,1}/\lambda = (1 + K^2 \omega^2)^{-\frac{1}{2}} \times (G_0/F) \\ &= G_0(F^2 + 4D^2 \omega^2)^{-\frac{1}{2}}. \end{aligned} \quad (41)$$

In this case, v_{crit} is seen to be smaller than for the case in which there is no nonlinearity in the conductor. For $F \rightarrow 0$,

$$v_{crit} = G_0/2D\omega. \quad (42)$$

• Step input

Consider an input signal $\varphi(t)$ of the form

$$\varphi(t) = v_0(1 + e^{-2at} - 2e^{-at}). \quad (43)$$

A signal of this form can be made to approximate rather well any realistic switching waveform with proper choice of a . Equation (43) provides us with the following inverse function:

$$\varphi^{-1}(v) = -\frac{1}{a} \ln \left(1 - \sqrt{\frac{v}{v_0}} \right). \quad (44)$$

From equation (28), we now have

$$\begin{aligned} \xi &= -\frac{1}{a} \ln \left\{ 1 - \left(\frac{e^{\beta x}(v/v_0)}{1 + \lambda v(1 - e^{\beta x})} \right)^{\frac{1}{2}} \right\} \\ &\quad - \frac{\gamma}{\beta \lambda} \ln \{ 1 + \lambda v(1 - e^{\beta x}) \}. \end{aligned} \quad (45)$$

Using the same notation as before, we have

$$a\xi = -\ln(1 - \sqrt{y}) + Ka \ln(1 + Zy). \quad (46)$$

By proceeding exactly as before, it can be shown that for $0 < Ka < 1/2$, the formation of shock waves is possible for either of the two conditions: (1) both F and $D > 0$; (2) both F and $D < 0$. However, for $Ka > 1/2$ or $Ka < 0$, $D < 0$ is a necessary condition for sharpening of wavefronts and formation of shock waves. In all cases the shock waves will start forming at

$$Z_{crit} = \frac{1 - 2Ka}{K^2 a^2}. \quad (47)$$

In Fig. 2, Eq. (46) has been plotted for $Ka = -1$ and for various values of Z . It is seen from this plot that, for an input of unit amplitude, $Z_{crit} = 3$ as given by (47). The curve for $Zv_0 = 4$ has no physical meaning but is drawn just to show the multiple-valued nature of the solution. Clearly the analysis is no longer valid.

As before, the critical amplitude of the input signal below which shock waves will not be formed in the line at any distance from the input is given by

$$v_{crit} = \frac{Z_{crit}}{\lambda} = \frac{G_0}{Da} \left[\frac{F}{4Da} - 1 \right]. \quad (48)$$

For $F \rightarrow 0$,

$$v_{crit} = -G_0/Da. \quad (49)$$

By considering various cases, it can easily be shown from the above analysis that when the nonlinear part of the conductance is such that $F > 0$, v_{crit} is larger and when $F < 0$, v_{crit} is smaller than for the case when there is no nonlinearity present in the conductor. It is assumed here of course that we are considering only the positive-going step input. This distinction did not appear in the previous case of a sinusoidal input for then the signal went through equal positive and negative excursions.

Conclusions

Using the perturbation technique, we have solved the problem of propagation of signals in a nonlinear transmission line with weak nonlinearity in its distributed capacitance and small nonlinear shunt loss. The conditions for the formation of shock waves in such a line are also derived. It is shown that for sinusoidal inputs, the presence of nonlinearity in the shunt loss makes possible the formation of shock waves at a smaller amplitude than when the nonlinearity is absent. For step-like inputs, this critical amplitude either increases or decreases depending on the sign of the nonlinearity and the polarity of the step. These results should prove useful for the study of small signal distortions in propagation through a nonlinear, dispersive medium.

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